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Delta-Nabla Isoperimetric Problems

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Abstract

We prove general necessary optimality conditions for delta-nabla isoperimetric problems of the calculus of variations.

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1 Introduction

Isoperimetric problems consist in maximizing or minimizing a cost functional subject to integral constraints. They have found a broad class of important applications throughout the centuries. Areas of application include astronomy, geometry, algebra, and analysis [4]. The study of isoperimetric problems is nowadays done, in an elegant and rigorously way, by means of the theory of the calculus of variations [18], and concrete isoperimetric problems in engineering have been investigated by a number of authors [9]. For recent developments on isoperimetric problems we refer the reader to [2, 1, 11] and references therein.

A new delta-nabla calculus of variations has recently been introduced by the authors in [14]. The new calculus of variations allow us to unify and extend the two standard approaches of the calculus of variations on time scales [10, 16, 17], and is motivated by applications in economics [8].

The delta-nabla variational theory is still in the very beginning, and much remains to be done. In this note we develop further the theory by introducing the isoperimetric problem in the delta-nabla setting and proving respective

necessary optimality conditions. Section 2 reviews the Euler-Lagrange equations of the delta-nabla calculus of variations [14] and recalls the results of the literature needed in the sequel. Our contribution is given in Section 3, where the delta-nabla isoperimetric problem is formulated and necessary optimality conditions for both normal and abnormal extremizers are proved (see Theorems 8 and 10). We proceed with Section 4, illustrating the applicability of our results with an example. Finally, we present the conclusion (Section 5) and some open problems (Section 6).

2 Preliminaries

We assume the reader to be familiar with the theory of time scales. For an introduction to the calculus on time scales we refer to the books [6, 7, 13].

Let $\mathbb{T}$ be a given time scale with jump operators σ and ρ, and differential operators Δ and ∇. Let $a, b \in \mathbb{T}$, $a < b$, and $(\mathbb{T} \setminus \{a, b\}) \cap [a, b] \neq \emptyset$; and $L_\Delta(\cdot,\cdot,\cdot)$ and $L_\nabla(\cdot,\cdot,\cdot)$ be two given smooth functions from $\mathbb{T} \times \mathbb{R}^2$ to $\mathbb{R}$. The results here discussed are trivially generalized for admissible functions $y : \mathbb{T} \to \mathbb{R}^n$ but for simplicity of presentation we restrict ourselves to the scalar case $n = 1$. Throughout the text we use the operators $[y]$ and $\{y\}$ defined by

$$[y](t) := \left(t, y^\sigma(t), y^\Delta(t)\right) , \quad \{y\}(t) := \left(t, y^\rho(t), y^\nabla(t)\right) .$$

In [14] the problem of extremizing a delta-nabla variational functional subject to given boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ is posed and studied:

$$\begin{gathered} \mathcal{J}(y) = \left(\int_a^b L_\Delta[y](t)\Delta t\right)\left(\int_a^b L_\nabla\{y\}(t)\nabla t\right) \longrightarrow \text{extr} \\ y \in C^1_\diamond\left([a,b],\mathbb{R}\right) \\ y(a) = \alpha\,, \quad y(b) = \beta\,, \end{gathered} \tag{1}$$

where $C^1_\diamond\left([a,b],\mathbb{R}\right)$ denote the class of functions $y : [a, b] \to \mathbb{R}$ with y^Δ continuous on $[a, b]^\kappa$ and y^∇ continuous on $[a, b]_\kappa$.

Definition 1. *We say that $\hat{y} \in C^1_\diamond([a, b], \mathbb{R})$ is a weak local minimizer (respectively weak local maximizer) for problem* (1) *if there exists $\delta > 0$ such that $\mathcal{J}(\hat{y}) \le \mathcal{J}(y)$ (respectively $\mathcal{J}(\hat{y}) \ge \mathcal{J}(y)$) for all $y \in C^1_\diamond([a, b], \mathbb{R})$ satisfying the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$, and $||y - \hat{y}||_{1,\infty} < \delta$, where $||y||_{1,\infty} := ||y^\sigma||_\infty + ||y^\rho||_\infty + ||y^\Delta||_\infty + ||y^\nabla||_\infty$ and $||y||_\infty := \sup_{t\in[a,b]^\kappa_\kappa} |y(t)|$.*

The main result of [14] gives two different forms for the Euler–Lagrange equation on time scales associated with the variational problem (1).

Theorem 2 (The general Euler-Lagrange equations on time scales [14]). *If $\hat{y} \in C_\diamond^1$ is a weak local extremizer of problem* (1)*, then $\hat{y}$ satisfies the following delta-nabla integral equations:*

$$
\begin{aligned}
\mathcal{J}_\nabla(\hat{y}) &\left(\partial_3 L_\Delta[\hat{y}](\rho(t)) - \int_a^{\rho(t)} \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau\right) \\
&+ \mathcal{J}_\Delta(\hat{y}) \left(\partial_3 L_\nabla\{\hat{y}\}(t) - \int_a^{t} \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau\right) = const \quad \forall t \in [a,b]_\kappa\,; \quad (2)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_\nabla(\hat{y}) &\left(\partial_3 L_\Delta[\hat{y}](t) - \int_a^{t} \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau\right) \\
&+ \mathcal{J}_\Delta(\hat{y}) \left(\partial_3 L_\nabla\{\hat{y}\}(\sigma(t)) - \int_a^{\sigma(t)} \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau\right) = const \quad \forall t \in [a,b]^\kappa\,. \quad (3)
\end{aligned}
$$

Remark 1. *In the classical context (i.e., when $\mathbb{T} = \mathbb{R}$) the necessary conditions* (2) *and* (3) *coincide with the Euler–Lagrange equations recently obtained in [8].*

Our main goal is to generalize Theorem 2 by covering variational problems subject to isoperimetric constraints. In order to do it (cf. proof of Theorem 8) we use some relationships of [3] between the delta and nabla derivatives, and some relationships of [12] between the delta and nabla integrals.

Proposition 3 (Theorems 2.5 and 2.6 of [3]). *(i) If $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^\kappa$ and f^Δ is continuous on $\mathbb{T}^\kappa$, then f is nabla differentiable on $\mathbb{T}_\kappa$ and*

$$f^\nabla(t) = \left(f^\Delta\right)^\rho(t) \quad \text{for all } t \in \mathbb{T}_\kappa\,. \quad (4)$$

(ii) If $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable on $\mathbb{T}_\kappa$ and f^∇ is continuous on $\mathbb{T}_\kappa$, then f is delta differentiable on $\mathbb{T}^\kappa$ and

$$f^\Delta(t) = \left(f^\nabla\right)^\sigma(t) \quad \text{for all } t \in \mathbb{T}^\kappa\,. \quad (5)$$

Proposition 4 (Proposition 7 of [12]). *If function $f : \mathbb{T} \to \mathbb{R}$ is continuous, then for all $a, b \in \mathbb{T}$ with $a < b$ we have*

$$\int_a^b f(t)\Delta t = \int_a^b f^\rho(t)\nabla t\,, \quad (6)$$

$$\int_a^b f(t)\nabla t = \int_a^b f^\sigma(t)\Delta t\,. \quad (7)$$

We also use the nabla Dubois–Reymond lemma of [16].

Lemma 5 (Lemma 14 of [16]). *Let $f \in C_{ld}([a,b],\mathbb{R})$. If*

$$\int_a^b f(t)\eta^{\nabla}(t)\nabla t = 0 \quad \text{for all } \eta \in C_{ld}^1([a,b],\mathbb{R}) \text{ with } \eta(a)=\eta(b)=0\,,$$

then $f(t)=c$ on $t\in[a,b]_\kappa$ for some constant c.

3 Main Results

We consider delta-nabla isoperimetric problems on time scales. The problem consists of extremizing

$$\mathcal{L}(y) = \left(\int_a^b L_\Delta[y](t)\Delta t\right)\left(\int_a^b L_\nabla\{y\}(t)\nabla t\right) \longrightarrow \text{extr} \tag{8}$$

in the class of functions $y \in C_\diamond^1([a,b],\mathbb{R})$ satisfying the boundary conditions

$$y(a)=\alpha\,, \quad y(b)=\beta\,, \tag{9}$$

and the constraint

$$\mathcal{K}(y) = \left(\int_a^b K_\Delta[y](t)\Delta t\right)\left(\int_a^b K_\nabla\{y\}(t)\nabla t\right) = k, \tag{10}$$

where α, β, k are given real numbers.

Definition 6. *We say that $\hat{y}\in C_\diamond^1([a,b],\mathbb{R})$ is a weak local minimizer (respectively weak local maximizer) for (8)–(10) if there exists $\delta>0$ such that*

$$\mathcal{L}(\hat{y}) \le \mathcal{L}(y) \quad (\text{respectively } \mathcal{L}(\hat{y}) \ge \mathcal{L}(y))$$

for all $y\in C_\diamond^1([a,b],\mathbb{R})$ satisfying the boundary conditions (9), the isoperimetric constraint (10), and $||y-\hat{y}||_{1,\infty}<\delta$.

Definition 7. *We say that $\hat{y}\in C_\diamond^1$ is an extremal for $\mathcal{K}$ if $\hat{y}$ satisfies the delta-nabla integral equations (2) and (3) for $\mathcal{K}$, i.e.,*

$$\begin{aligned}\mathcal{K}_\nabla(\hat{y})&\left(\partial_3 K_\Delta[\hat{y}](\rho(t)) - \int_a^{\rho(t)} \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right)\\ &+\mathcal{K}_\Delta(\hat{y})\left(\partial_3 K_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right) = const \quad \forall t\in[a,b]_\kappa\,;\end{aligned} \tag{11}$$

$$\mathcal{K}_{\nabla}(\hat{y})\left(\partial_3 K_{\Delta}[\hat{y}](t)-\int_a^t \partial_2 K_{\Delta}[\hat{y}](\tau)\Delta\tau\right)$$
$$+\mathcal{K}_{\Delta}(\hat{y})\left(\partial_3 K_{\nabla}\{\hat{y}\}(\sigma(t))-\int_a^{\sigma(t)} \partial_2 K_{\nabla}\{\hat{y}\}(\tau)\nabla\tau\right)=const \quad \forall t\in[a,b]^{\kappa}\,. \tag{12}$$

An extremizer (i.e., a weak local minimizer or a weak local maximizer) for the problem (8)–(10) *that is not an extremal for* $\mathcal{K}$ *is said to be a normal extremizer; otherwise (i.e., if it is an extremal for* $\mathcal{K}$*), the extremizer is said to be abnormal.*

Theorem 8. *If* $\hat{y}\in C^1_{\diamond}([a,b],\mathbb{R})$ *is a normal extremizer for the isoperimetric problem* (8)–(10)*, then there exists* $\lambda\in\mathbb{R}$ *such that* $\hat{y}$ *satisfies the following delta-nabla integral equations:*

$$\mathcal{L}_{\nabla}(\hat{y})\left(\partial_3 L_{\Delta}[\hat{y}](\rho(t))-\int_a^{\rho(t)} \partial_2 L_{\Delta}[\hat{y}](\tau)\Delta\tau\right)$$
$$+\mathcal{L}_{\Delta}(\hat{y})\left(\partial_3 L_{\nabla}\{\hat{y}\}(t)-\int_a^{t} \partial_2 L_{\nabla}\{\hat{y}\}(\tau)\nabla\tau\right)$$
$$-\lambda\left\{\mathcal{K}_{\nabla}(\hat{y})\left(\partial_3 K_{\Delta}[\hat{y}](\rho(t))-\int_a^{\rho(t)} \partial_2 K_{\Delta}[\hat{y}](\tau)\Delta\tau\right)\right.$$
$$\left.+\mathcal{K}_{\Delta}(\hat{y})\left(\partial_3 K_{\nabla}\{\hat{y}\}(t)-\int_a^{t} \partial_2 K_{\nabla}\{\hat{y}\}(\tau)\nabla\tau\right)\right\}=const \quad \forall t\in[a,b]_{\kappa}\,; \tag{13}$$

$$\mathcal{L}_{\nabla}(\hat{y})\left(\partial_3 L_{\Delta}[\hat{y}](t)-\int_a^{t} \partial_2 L_{\Delta}[\hat{y}](\tau)\Delta\tau\right)$$
$$+\mathcal{L}_{\Delta}(\hat{y})\left(\partial_3 L_{\nabla}\{\hat{y}\}(\sigma(t))-\int_a^{\sigma(t)} \partial_2 L_{\nabla}\{\hat{y}\}(\tau)\nabla\tau\right)$$
$$-\lambda\left\{\mathcal{K}_{\nabla}(\hat{y})\left(\partial_3 K_{\Delta}[\hat{y}](t)-\int_a^{t} \partial_2 K_{\Delta}[\hat{y}](\tau)\Delta\tau\right)\right.$$
$$\left.+\mathcal{K}_{\Delta}(\hat{y})\left(\partial_3 K_{\nabla}\{\hat{y}\}(\sigma(t))-\int_a^{\sigma(t)} \partial_2 K_{\nabla}\{\hat{y}\}(\tau)\nabla\tau\right)\right\}=const \quad \forall t\in[a,b]^{\kappa}\,. \tag{14}$$

Proof. Consider a variation of $\hat{y}$, say $\bar{y}=\hat{y}+\varepsilon_1\eta_1+\varepsilon_2\eta_2$, where for each $i\in\{1,2\}$, $\eta_i\in C^1_{\diamond}([a,b],\mathbb{R})$ and $\eta_i(a)=\eta_i(b)=0$, and ε_i is a sufficiently small parameter (ε_1 and ε_2 must be such that $||\bar{y}-\hat{y}||_{1,\infty}<\delta$ for some $\delta>0$). Here,

η_1 is an arbitrary fixed function and η_2 is a fixed function that will be chosen later. Define the real function

$$\bar{K}(\varepsilon_1,\varepsilon_2)=\mathcal{K}(\bar{y})=\left(\int_a^b K_\Delta[\bar{y}](t)\Delta t\right)\left(\int_a^b K_\nabla\{\bar{y}\}(t)\nabla t\right)-k.$$

We have

$$\begin{aligned}\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} &= \mathcal{K}_\nabla(\hat{y})\int_a^b \left(\partial_2 K_\Delta[\hat{y}](t)\eta_2^\sigma(t)+\partial_3 K_\Delta[\hat{y}](t)\eta_2^\Delta(t)\right)\Delta t\\ &\quad + \mathcal{K}_\Delta(\hat{y})\int_a^b \left(\partial_2 K_\nabla\{\hat{y}\}(t)\eta_2^\rho(t)+\partial_3 K_\nabla\{\hat{y}\}(t)\eta_2^\nabla(t)\right)\nabla t = 0\,.\end{aligned}$$

We now make use of the following formulas of integration by parts [6]: if functions $f,g:\mathbb{T}\to\mathbb{R}$ are delta and nabla differentiable with continuous derivatives, then

$$\begin{aligned}\int_a^b f^\sigma(t)g^\Delta(t)\Delta t &= (fg)(t)|_{t=a}^{t=b}-\int_a^b f^\Delta(t)g(t)\Delta t\,,\\ \int_a^b f^\rho(t)g^\nabla(t)\nabla t &= (fg)(t)|_{t=a}^{t=b}-\int_a^b f^\nabla(t)g(t)\nabla t\,.\end{aligned}$$

Having in mind that $\eta_2(a)=\eta_2(b)=0$, we obtain:

$$\begin{aligned}&\int_a^b \partial_2 K_\Delta[\hat{y}](t)\eta_2^\sigma(t)\Delta t = \int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\,\eta_2(t)|_{t=a}^{t=b}\\ &-\int_a^b\left(\int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right)\eta_2^\Delta(t)\Delta t = -\int_a^b\left(\int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right)\eta_2^\Delta(t)\Delta t\end{aligned}$$

and

$$\begin{aligned}&\int_a^b \partial_2 K_\nabla\{\hat{y}\}(t)\eta_2^\rho(t)\nabla t = \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\,\eta_2(t)|_{t=a}^{t=b}\\ &-\int_a^b\left(\int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right)\eta_2^\nabla(t)\nabla t = -\int_a^b\left(\int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right)\eta_2^\nabla(t)\nabla t.\end{aligned}$$

Therefore,

$$\begin{aligned}\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} &= \mathcal{K}_\nabla(\hat{y})\int_a^b\left(\partial_3 K_\Delta[\hat{y}](t)-\int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right)\eta_2^\Delta(t)\Delta t\\ &\quad + \mathcal{K}_\Delta(\hat{y})\int_a^b\left(\partial_3 K_\nabla\{\hat{y}\}(t)-\int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right)\eta_2^\nabla(t)\nabla t. \quad (15)\end{aligned}$$

Let

$$f(t) = \mathcal{K}_\nabla(\hat{y}) \left(\partial_3 K_\Delta[\hat{y}](t) - \int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau \right)$$

and

$$g(t) = \mathcal{K}_\Delta(\hat{y}) \left(\partial_3 K_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau \right).$$

We can then write equation (15) in the form

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} = \int_a^b f(t)\eta_2^\Delta(t)\Delta t + \int_a^b g(t)\eta_2^\nabla(t)\nabla t. \tag{16}$$

Transforming the delta integral in (16) to a nabla integral by means of (6) we obtain

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} = \int_a^b f^\rho(t)(\eta_2^\Delta)^\rho(t)\nabla t + \int_a^b g(t)\eta_2^\nabla(t)\nabla t$$

and by (4)

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} = \int_a^b \left(f^\rho(t) + g(t)\right)\eta_2^\nabla(t)\nabla t.$$

As $\hat{y}$ is a normal extremizer we conclude, by Lemma 5 and equation (12), that there exists η_2 such that $\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} \neq 0$. Since $\bar{K}(0,0) = 0$, by the implicit function theorem we conclude that there exists a function ε_2 defined in the neighborhood of zero, such that $\bar{K}(\varepsilon_1, \varepsilon_2(\varepsilon_1)) = 0$, i.e., we may choose a subset of variations $\bar{y}$ satisfying the isoperimetric constraint.

Let us now consider the real function

$$\bar{L}(\varepsilon_1, \varepsilon_2) = \mathcal{L}(\bar{y}) = \left(\int_a^b L_\Delta[\bar{y}](t)\Delta t \right) \left(\int_a^b L_\nabla\{\bar{y}\}(t)\nabla t \right).$$

By hypothesis, $(0,0)$ is an extremal of $\bar{L}$ subject to the constraint $\bar{K} = 0$ and $\nabla\bar{K}(0,0) \neq \mathbf{0}$. By the Lagrange multiplier rule, there exists some real λ such that $\nabla(\bar{L}(0,0) - \lambda\bar{K}(0,0)) = \mathbf{0}$. Having in mind that $\eta_1(a) = \eta_1(b) = 0$, we can write

$$\begin{aligned} \left.\frac{\partial \bar{L}}{\partial \varepsilon_1}\right|_{(0,0)} &= \mathcal{L}_\nabla(\hat{y}) \int_a^b \left(\partial_3 L_\Delta[\hat{y}](t) - \int_a^t \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau \right) \eta_1^\Delta(t)\Delta t \\ &\quad + \mathcal{L}_\Delta(\hat{y}) \int_a^b \left(\partial_3 L_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau \right) \eta_1^\nabla(t)\nabla t \end{aligned} \tag{17}$$

and

$$\begin{aligned} \left.\frac{\partial \bar{K}}{\partial \varepsilon_1}\right|_{(0,0)} &= \mathcal{K}_\nabla(\hat{y}) \int_a^b \left(\partial_3 K_\Delta[\hat{y}](t) - \int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau \right) \eta_1^\Delta(t)\Delta t \\ &\quad + \mathcal{K}_\Delta(\hat{y}) \int_a^b \left(\partial_3 K_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau \right) \eta_1^\nabla(t)\nabla t. \end{aligned} \tag{18}$$

Let

$$m(t)=\mathcal{L}_{\nabla}(\hat{y})\left(\partial_3 L_{\Delta}[\hat{y}](t)-\int_a^t \partial_2 L_{\Delta}[\hat{y}](\tau)\Delta\tau\right)$$

and

$$n(t)=\mathcal{L}_{\Delta}(\hat{y})\left(\partial_3 L_{\nabla}\{\hat{y}\}(t)-\int_a^t \partial_2 L_{\nabla}\{\hat{y}\}(\tau)\nabla\tau\right).$$

Then equations (17) and (18) can be written in the form

$$\left.\frac{\partial \bar{L}}{\partial \varepsilon_1}\right|_{(0,0)}=\int_a^b m(t)\eta_1^{\Delta}(t)\Delta t+\int_a^b n(t)\eta_1^{\nabla}(t)\nabla t$$

and

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_1}\right|_{(0,0)}=\int_a^b f(t)\eta_1^{\Delta}(t)\Delta t+\int_a^b g(t)\eta_1^{\nabla}(t)\nabla t.$$

Transforming the delta integrals in the above equalities to nabla integrals by means of (6) and using (4) we obtain

$$\left.\frac{\partial \bar{L}}{\partial \varepsilon_1}\right|_{(0,0)}=\int_a^b \left(m^{\rho}(t)+n(t)\right)\eta_1^{\nabla}(t)\nabla t$$

and

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_1}\right|_{(0,0)}=\int_a^b \left(f^{\rho}(t)+g(t)\right)\eta_1^{\nabla}(t)\nabla t.$$

Therefore,

$$\int_a^b \eta_1^{\Delta}(t)\left\{m^{\rho}(t)+n(t)-\lambda\left(f^{\rho}(t)+g(t)\right)\right\}\nabla t=0. \tag{19}$$

Since (19) holds for any η_1, by Lemma 5 we have

$$m^{\rho}(t)+n(t)-\lambda\left(f^{\rho}(t)+g(t)\right)=c$$

for some $c\in\mathbb{R}$ and all $t\in[a,b]_{\kappa}$. Hence, condition (13) holds. In a similar way we can obtain equation (14). In that case we use relationships (5) and (7), and [5, Lemma 4.1]. □

In the particular case $L_{\nabla}\equiv\frac{1}{b-a}$ we get from Theorem 8 the main result of [11]:

Corollary 9 (Theorem 3.4 of [11])**.** *Suppose that*

$$J(y)=\int_a^b L(t,y^{\sigma}(t),y^{\Delta}(t))\Delta t$$

has a local minimum at y_* subject to the boundary conditions $y(a) = y_a$ and $y(b) = y_b$ and the isoperimetric constraint

$$I(y) = \int_a^b g(t, y^\sigma(t), y^\Delta(t))\Delta t = k\,.$$

Assume that y_* is not an extremal for the functional I. Then, there exists a Lagrange multiplier constant λ such that y_* satisfies the following equation:

$$\partial_3 F^\Delta(t, y_*^\sigma(t), y_*^\Delta(t)) - \partial_2 F(t, y_*^\sigma(t), y_*^\Delta(t)) = 0 \quad \textit{for all} \quad t \in [a,b]^{\kappa^2},$$

where $F = L - \lambda g$ and $\partial_3 F^\Delta$ denotes the delta derivative of a composition.

One can easily cover abnormal extremizers within our result by introducing an extra multiplier λ_0.

Theorem 10. *If $\hat{y} \in C^1_\diamond$ is an extremizer for the isoperimetric problem* (8)–(10), *then there exist two constants λ_0 and λ, not both zero, such that $\hat{y}$ satisfies the following delta-nabla integral equations:*

$$\begin{aligned}\lambda_0 \Bigg\{ \mathcal{L}_\nabla(\hat{y}) \left(\partial_3 L_\Delta[\hat{y}](\rho(t)) - \int_a^{\rho(t)} \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau \right) \\ + \mathcal{L}_\Delta(\hat{y}) \left(\partial_3 L_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau \right) \Bigg\} \\ - \lambda \Bigg\{ \mathcal{K}_\nabla(\hat{y}) \left(\partial_3 K_\Delta[\hat{y}](\rho(t)) - \int_a^{\rho(t)} \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau \right) \\ + \mathcal{K}_\Delta(\hat{y}) \left(\partial_3 K_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau \right) \Bigg\} = const \quad \forall t \in [a,b]_\kappa\,; \end{aligned} \tag{20}$$

$$\begin{aligned}\lambda_0 \Bigg\{ \mathcal{L}_\nabla(\hat{y}) \left(\partial_3 L_\Delta[\hat{y}](t) - \int_a^t \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau \right) \\ + \mathcal{L}_\Delta(\hat{y}) \left(\partial_3 L_\nabla\{\hat{y}\}(\sigma(t)) - \int_a^{\sigma(t)} \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau \right) \Bigg\} \\ - \lambda \Bigg\{ \mathcal{K}_\nabla(\hat{y}) \left(\partial_3 K_\Delta[\hat{y}](t) - \int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau \right) \\ + \mathcal{K}_\Delta(\hat{y}) \left(\partial_3 K_\nabla\{\hat{y}\}(\sigma(t)) - \int_a^{\sigma(t)} \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau \right) \Bigg\} = const \quad \forall t \in [a,b]^\kappa\,. \end{aligned} \tag{21}$$

Proof. Following the proof of Theorem 8, since $(0,0)$ is an extremal of $\bar{L}$ subject to the constraint $\bar{K}=0$, the extended Lagrange multiplier rule (see for instance [18, Theorem 4.1.3]) asserts the existence of reals λ_0 and λ, not both zero, such that $\nabla(\lambda_0\bar{L}(0,0)-\lambda\bar{K}(0,0))=\mathbf{0}$. Therefore,

$$\int_a^b \eta_1^\Delta(t)\left\{\lambda_0\left(m^\rho(t)+n(t)\right)-\lambda\left(f^\rho(t)+g(t)\right)\right\}\nabla t=0. \tag{22}$$

Since (22) holds for any η_1, by Lemma 5, we have

$$\lambda_0\left(m^\rho(t)+n(t)\right)-\lambda\left(f^\rho(t)+g(t)\right)=c$$

for some $c\in\mathbb{R}$ and all $t\in[a,b]_\kappa$. This establishes equation (20). Equation (21) can be shown using a similar technique. □

Remark 2. *If $\hat{y}\in C_\diamond^1$ is an extremizer for the isoperimetric problem* (8)–(10)*, then we can choose $\lambda_0=1$ in Theorem 10 and obtain Theorem 8. For abnormal extremizers, Theorem 10 holds with $\lambda_0=0$. The condition $(\lambda_0,\lambda)\neq$ $\mathbf{0}$ guarantees that Theorem 10 is a useful necessary optimality condition.*

In the particular case $L_\Delta\equiv\frac{1}{b-a}$ we get from Theorem 10 the main result of [2]:

Corollary 11 (Theorem 2 of [2])**.** *If y is a local minimizer or maximizer for*

$$I[y]=\int_a^b f(t,y^\rho(t),y^\nabla(t))\nabla t$$

subject to the boundary conditions $y(a)=\alpha$ and $y(b)=\beta$ and the nabla-integral constraint

$$J[y]=\int_a^b g(t,y^\rho(t),y^\nabla(t))\nabla t=\Lambda\,,$$

then there exist two constants λ_0 and λ, not both zero, such that

$$\partial_3K^\nabla\left(t,y^\rho(t),y^\nabla(t)\right)-\partial_2K\left(t,y^\rho(t),y^\nabla(t)\right)=0$$

for all $t\in[a,b]_\kappa$, where $K=\lambda_0 f-\lambda g$.

4 An Example

Let $\mathbb{T}=\{1,2,3,\ldots,M\}$, where $M\in\mathbb{N}$ and $M\geq 2$. Consider the problem

$$\text{minimize}\quad \mathcal{L}(y)=\left(\int_0^M (y^\Delta(t))^2\Delta t\right)\left(\int_0^M\left(y^\nabla(t))^2+y^\nabla(t)\right)\nabla t\right) \tag{23}$$
$$y(0)=0,\quad y(M)=M,$$

subject to the constraint

$$\mathcal{K}(y)=\int_0^M t y^{\Delta}(t)\Delta t = 1. \tag{24}$$

Since

$$L_{\Delta}=(y^{\Delta})^2, \quad L_{\nabla}=(y^{\nabla})^2+y^{\nabla}, \quad K_{\Delta}=ty^{\Delta}, \quad K_{\nabla}=\frac{1}{M}$$

we have

$$\partial_2 L_{\Delta}=0, \quad \partial_3 L_{\Delta}=2y^{\Delta}, \quad \partial_2 L_{\nabla}=0, \quad \partial_3 L_{\nabla}=2y^{\nabla}+1,$$

and

$$\partial_2 K_{\Delta}=0, \quad \partial_3 K_{\Delta}=t, \quad \partial_2 K_{\nabla}=0, \quad \partial_3 K_{\nabla}=0.$$

As

$$\mathcal{K}_{\nabla}(\hat{y})\left(\partial_3 K_{\Delta}[\hat{y}](t)-\int_a^t \partial_2 K_{\Delta}[\hat{y}](\tau)\Delta\tau\right) + \mathcal{K}_{\Delta}(\hat{y})\left(\partial_3 K_{\nabla}\{\hat{y}\}(\sigma(t))-\int_a^{\sigma(t)} \partial_2 K_{\nabla}\{\hat{y}\}(\tau)\nabla\tau\right)=t$$

there are no abnormal extremals for the problem (23)–(24). Applying equation (14) of Theorem 8 we get the following delta-nabla differential equation:

$$2Ay^{\Delta}(t)+B+2By^{\nabla}(\sigma(t))-\lambda t=C, \tag{25}$$

where $C\in\mathbb{R}$ and A, B are the values of functionals $\mathcal{L}_{\nabla}$ and $\mathcal{L}_{\Delta}$ in a solution of (23)–(24), respectively. Since $y^{\nabla}(\sigma(t))=y^{\Delta}(t)$ (5), we can write equation (25) in the form

$$2Ay^{\Delta}(t)+B+2By^{\Delta}-\lambda t=C. \tag{26}$$

Observe that $B\neq 0$ and $A>2$. Hence, solving equation (26) subject to the boundary conditions $y(0)=0$ and $y(M)=M$ we get

$$y(t)=\left[1-\frac{\lambda\,(M-t)}{4(A+B)}\right]t\,. \tag{27}$$

Substituting (27) into (24) we obtain $\lambda=-\frac{(A+B)(M-2)}{12M(M-1)}$. Hence,

$$y(t)=\frac{(4\,M^2-7\,M-3M\,t+6\,t)\,t}{M\,(M-1)}$$

is an extremal for the problem (23)–(24).

5 Conclusion

Minimization of functionals given by the product of two integrals were considered by Euler himself, and are now receiving an increase of interest due to their nonlocal properties and applications to economics [8, 14]. In this paper we obtained general necessary optimality conditions for isoperimetric problems of the calculus of variations on time scales. Our results extend the ones with delta derivatives proved in [11] and analogous nabla results [2] to more general variational problems described by the product of delta and nabla integrals.

6 Open Problems

The results here obtained can be generalized in different ways: (i) to variational problems involving higher-order delta and nabla derivatives, unifying and extending the higher-order results on time scales of [10] and [16]; (ii) to problems of the calculus of variations with a functional which is the composition of a certain scalar function H with the delta integral of a vector valued field f_Δ and a nabla integral of a vector field f_∇, i.e., of the form

$$H\left(\int_a^b f_\Delta(t, y^\sigma(t), y^\Delta(t))\Delta t\,, \int_a^b f_\nabla(t, y^\rho(t), y^\nabla(t))\nabla t\right) .$$

It remains to prove Euler-Lagrange equations and natural boundary conditions for such problems on time scales, with or without constraints.

Sufficient optimality conditions for delta-nabla problems of the calculus of variations is a completely open question. It would be also interesting to study direct optimization methods, extending the results of [15] to the more general delta-nabla setting.

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